Limit distributions for multitype branching processes of m-ary search trees [1]

by

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Abstract. A particular continuous-time multitype branching process is considered, it is the continuous-time embedding of a discrete-time process which is very popular in theoretical computer science: the m-ary search tree (m is an integer). There is a well-known phase transition: when $m \leq 26$, the asymptotic behavior of the process is Gaussian, but for $m \geq 27$ it is no more Gaussian and a limit W of a complex-valued martingale arises. Thanks to the branching property it appears as a solution of a *smoothing* equation of the type $Z \overset{\mathcal{L}}{=} e^{-\lambda T}(Z^{(1)} + \cdots + Z^{(m)})$, where $\lambda \in \mathbb{C}$, the $Z^{(k)}$ are independent copies of Z and T is a $\mathbb{R}_+$-valued random variable, independent of the $Z^{(k)}$. This distributional equation is extensively studied by various approaches. The existence and unicity of solution of the equation are proved by contraction methods. The fact that the distribution of W is absolutely continuous and that its support is the whole complex plane is shown via Fourier analysis. Finally, the existence of exponential moments of W is obtained by considering W as the limit of a complex Mandelbrot cascade.

Contents

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1 Introduction

Consider a continuous-time multitype branching process $(X(t), t \geq 0)$. Types are seen as colors of particles and there are $m-1$ colors, where $m \geq 2$ is an integer. The reproduction of the process is given by a particular matrix R (written in (2.1)), and any particle of colour j lives a random time of exponential distribution with parameter j. Such a classical process is considered for example in Athreya and Ney [1] or Janson [12] and it is precisely defined in Section 2.

When it is stopped at the n-th jump time, this process is nothing but the composition vector process $(X_n^{DT}, n \geq 0)$ say, of an m-ary search tree, which is an important algorithmic structure in computer science. A numerous literature is devoted to the asymptotic behavior of this composition vector. A famous phase transition appears. When $m \leq 26$, the random vector admits a central limit theorem with convergence in distribution to a Gaussian vector: see Mahmoud and Pittel [18], Lew and Mahmoud [13]. When $m \geq 27$, it has been proved that

$$X_n^{DT} = nv_1 + \Re(n^{\lambda_2} W^{DT} v_2) + o(n^{\sigma_2}) \quad a.s.,$$

where λ_2 is a complex number having a real part in $]\frac{1}{2}, 1[$ (it is an eigenvalue of the replacement matrix R), where v_1, v_2 are deterministic vectors and W^{DT} is

the limit of a complex-valued martingale. Heated conjectures about the random variable W^{DT} remain open (see [4], [20], Chern and Hwang [6], Mahmoud [17], Janson [12]).

This article is focused on the asymptotic behavior of the continuous-time process $(X(t))$. The links between continuous-time and discrete-time processes are detailed in Section 4. In particular, we give the so-called *martingale connection* that relates the almost sure limits of both processes.

Inspired by the methods used for a two-color Pólya urn in [5], we first prove in Section 3 that for $m \geq 27$, $X(t)$ admits the following asymptotic expansion:

$$X(t) = e^t \xi v_1(1 + o(1)) + \Re(e^{\lambda_2 t} W v_2)(1 + o(1)) \ \text{ a.s. and in } L^p \ \ \forall p \geq 1,$$

where ξ is a Gamma distributed random variable and W a $\mathbb{C}$-valued one.

We are interested in the limit random variables $W_k, k = 1, \ldots, m-1$, each corresponding to $X_k(t)$ which denotes the process $X(t)$ when it starts from one particle of color k. Using the branching property, a system of dislocation equations is written for the random vectors $X_k(t)$ in Section 5.1. A system of fixed point equations satisfied by the corresponding limit laws is then derived in Section 5.2. In particular, the complex-valued random variable W_1 is a solution of the fixed point equation

$$Z \overset{\mathcal{L}}{=} e^{-\lambda_2 T}(Z^{(1)} + \cdots + Z^{(m)}), \tag{1.1}$$

where $\{Z^{(k)} : k \geq 1\}$ are independent copies of Z, $T = \tau_{(1)} + \cdots + \tau_{(m-1)}$, $\{\tau_{(j)} : j \geq 1\}$ are random variables independent of each other and independent of $\{Z^{(k)}\}$, each $\tau_{(j)}$ has distribution $\mathcal{E}xp(j)$ (we denote by $\mathcal{E}xp(j)$ the exponential distribution of parameter j: it has density $x \mapsto je^{-jx}$ on $]0, \infty[$).

Further properties of W_1 are derived from a fine study of Eq. (1.1). We first show in Theorems 6.12 and 6.14 that Eq. (1.1) admits a unique square-integrable solution having a given mean. In particular, this implies that Eq. (1.1) characterizes the distribution of W_1. This result is proven by two contraction methods applied to the corresponding smoothing transformations. The first one deals with suitable spaces of probability measures where the classical Wasserstein metric is adapted to the complex field; it leads to Theorem 6.12. The second contraction method, that gives a proof for Theorem 6.14, consists in working on Fourier transforms of solutions and provides a somehow simpler proof. Furthermore, this second method gives a result of existence and unicity for solutions of the convolution equation

$$\Phi(t) = \int_0^{+\infty} \Phi^m(t-u) f_T(u) du, \qquad t \in \mathbb{C},$$

in a convenient space of functions, where f_T denotes the density of T (see Remark 6.16).

Once the characterization of W_1 by Eq. (1.1) is proven, it suffices to derive properties of solutions of this distributional equation. We show in this way the following results on the law of W_1.

Theorem 1.1 *When $m \geq 27$, the complex-valued random variable W_1 admits a density and its support is the whole complex plane. Its Fourier transform satisfies*

$$\mathbb{E}e^{i\langle t, W_1\rangle} = O(|t|^{-a})$$

when $|t| \to +\infty$, for some $a > 1$.

This theorem is a direct consequence of Theorem 7.17 that provides such properties for solutions of (1.1) admitting a nonzero mean. Our proof consists in showing successively that the characteristic function of any solution has modulus equal to 1 only at the origin, that it tends to zero at infinity, and finally that it is of order $O(|t|^{-a})$ as $|t| \to \infty$ for some $a > 1$ so that it is square-integrable on $\mathbb{C}$. In the approach we need to prove a non-lattice property of Eq. (1.1) *via* Gelfand-Schneider theorem, using the algebraicity of λ_2 (see proof of Lemma 7.20).

Theorem 1.2 *When $m \geq 27$, the random variable W_1 admits exponential moments in a neighbourhood of the origin of the complex plane. If $L_1(z) = \mathbb{E}e^{zW_1}$ denotes its Laplace series, then L_1 is holomorphic near 0 and, after a change of variable, the function $z \mapsto -\frac{\varrho}{z}L_1\left(z^{-\lambda_2}\right)$ is a solution of the differential equation*

$$y^{(m-1)} = y^m.$$

Theorem 1.2 is immediately derived from Theorems 8.27 and 8.30 just as Theorem 1.1 was derived from Theorem 7.17. To prove Theorems 8.27 and 8.30, we consider a solution of (1.1) as the limit of a complex Mandelbrot cascade. The results are a consequence of fine analytical properties of the Fourier transform of the limit variable.

The paper is organized as follows.

The continuous-time multitype branching process is defined in Section 2. Its relation with the m-ary search tree is detailed in Section 3, while Section 4 is devoted to the second order asymptotic expansion of $(X(t))_{t\geq 0}$ and to its connection with the corresponding discrete process. In Section 5, we use the branching property of the process to show that the martingale limits of the continuous-time process are related by a system of equations in law so that the fixed point

equation (1.1) emerges. These first four sections constitute the first part of the paper.

The second part of the paper consists in putting the focus on Eq. (1.1) that turns out to characterize the distribution of W_1 so that all results on solutions provide results on W_1. In Section 6 we define the natural smoothing transform associated with Eq. (1.1) and we show that it defines a contraction in the space of square-integrable probability measures with given mean. Results on the support and on absolute continuity of solutions are obtained in Section 7. Finally, Section 8 is devoted to the exponential moments and the Laplace series of solutions.

2 Definition of the branching process

In this section we introduce the definition of the continuous time multitype branching process $(X(t))$, and present the spectral decomposition of its transition matrix.

2.1 Infinitesimal generator

In the whole paper, the underlying vector space is $\mathbb{R}^{m-1}$ or sometimes $\mathbb{C}^{m-1}$. Let R be the following square matrix of order $m-1$:

$$R = \begin{pmatrix} -1 & 1 & & & \\ & -1 & 1 & & \\ & & -1 & \ddots & \\ & & & \ddots & 1 \\ m & & & & -1 \end{pmatrix}, \tag{2.1}$$

and for $k = 1, \ldots, m-1$, let w_k be the k-th row vector of R: when $1 \leq k \leq m-2$, the k-th coordinate of w_k equals -1, the $(k+1)$-th equals 1 and all the others are 0; w_{m-1} has m as first coordinate, -1 as last one, and 0 for all others.

Let G be the operator defined on functions f from $\mathbb{C}^{m-1}$ to any real or complex vector space by the following formula: for any vector v in $\mathbb{C}^{m-1}$,

$$G(f)(v) = \sum_{k=1}^{m-1} k l_k(v)[f(v + w_k) - f(v)], \tag{2.2}$$

where l_k are the coordinate forms: $l_k(x_1, \ldots, x_{m-1}) = x_k$.

Definition 2.3 *The right-continuous process $X = (X(t), t \geq 0)$ is the only continuous time Markov process with state space $\mathbb{R}^{m-1}$ having G as infinitesimal generator.*

Equivalently, X is a continuous time multitype branching process with $m-1$ types (or colors), having R as reproduction matrix. The k-th coordinate of the vector $X(t)$, namely $l_k(X(t))$, is the number of particles of color k at time t. A particle of color k lives a random exponential time with parameter k; when it dies, it reproduces one particle of color $k+1$ if $k = 1, \ldots, m-2$, and m particles of color 1 if $k = m-1$.

This branching continuous time process can be thought as the embedded process of a discrete Markov chain $X^{DT} = (X_n^{DT})_{n \in \mathbb{N}}$ which is a Pólya-type discrete Markov chain associated with the node process of an m-ary search tree, an important algorithmic structure in computer science. This connection is detailed in Section 3.

2.2 Spectral decomposition

Let R_G be the matrix of G's restriction to linear forms in the canonical basis $(l_k)_{1 \leq k \leq m-1}$. One immediately checks that

$$R_G = \begin{pmatrix} -1 & 1 & & & \\ & -2 & 2 & & \\ & & -3 & \ddots & \\ & & & \ddots & m-2 \\ m(m-1) & & & & -(m-1) \end{pmatrix},$$

where an empty entry means a zero entry. It has been established in many papers – see for example Mahmoud [17], Chern and Hwang [6] or [4] – and it can be easily checked that R_G's (unitary) characteristic polynomial is

$$\chi_{R_G}(\lambda) = \prod_{k=1}^{m-1} (\lambda + k) - m! = \frac{\Gamma(\lambda + m)}{\Gamma(\lambda + 1)} - m!, \tag{2.3}$$

where Γ denotes Euler's Gamma function. All eigenvalues are simple, 1 being the one having the largest real part.

In the whole paper, λ_2 will denote χ_{R_G}'s root having the second largest real part and a positive imaginary part.

The famous phase transition on m-ary search trees already mentioned in the introduction is due to the fact that

$$\Re(\lambda_2) > 1/2 \ \text{ if and only if } \ m \geq 27.$$

See for example [4]. The assumption $\Re(\lambda_2) > 1/2$ will be frequently used in the sequel.

We adopt the following notations:

$$\begin{cases} \forall n \in \mathbb{Z}_{\geq 0}, \ \binom{z}{n} = \dfrac{\Gamma(z+1)}{n!\Gamma(z-n+1)} = \dfrac{z(z-1)\ldots(z-n+1)}{n!}; \\ H_m(z) = \displaystyle\sum_{1\leq k\leq m-1} \frac{1}{z+k}; \\ u_1(x_1,\ldots,x_{m-1}) = \displaystyle\sum_{1\leq k\leq m-1} kx_k; \\ u_2(x_1,\ldots,x_{m-1}) = \displaystyle\sum_{1\leq k\leq m-1} \binom{\lambda_2+k-1}{k-1} x_k; \\ v_1 = \dfrac{1}{H_m(1)} \left(\dfrac{1}{k(k+1)}\right)_{1\leq k\leq m-1}; \\ v_2 = \dfrac{1}{H_m(\lambda_2)} \left(\dfrac{1}{k\binom{\lambda_2+k}{k}}\right)_{1\leq k\leq m-1}. \end{cases} \tag{2.4}$$

The linear forms u_1 and u_2 are eigenvectors of G, namely $G(u_1) = u_1$ and $G(u_2) = \lambda_2 u_2$. The vectors v_1 and v_2 are left eigenvectors of R_G, respectively associated with the eigenvalues 1 and λ_2. They satisfy $u_1(v_1) = u_2(v_2) = 1$ and $u_1(v_2) = u_2(v_1) = 0$. These eigendata had already been essentially computed in [4] and [20]. Note that, since λ_2 is not real, u_2 and v_2 have nonreal coordinates.

3 m-ary search trees and embedding

In this section we present the connection between m-ary search trees and multi-type branching processes. This example of embedding of a discrete time process into a continuous time process has already been evoked by Janson in [12].

3.1 m-ary search trees

We define here a discrete time Markov chain $X^{DT} = (X_n^{DT}, n \geq 0)$ with values in $\mathbb{N}^{m-1} \setminus \{0\}$. The i-th coordinate of X_n^{DT} is denoted by $X_n^{(i)}$ and has a "physical"

meaning detailed hereafter. The Markov chain X^{DT} is a random walk defined by an initial vector X_0^{DT} in $\mathbb{N}^{m-1} \setminus \{0\}$ and by the following transition probabilities: $\forall v \in \mathbb{N}^{m-1} \setminus \{0\}$, $\forall k = 1, \ldots, m-1$,

$$q(v, v + w_k) = \frac{k l_k(v)}{\sum_{j=1}^{m-1} j l_j(v)}, \tag{3.1}$$

where the increment vectors w_k are given in Section 2.1 and $l_k(v)$ denotes the k-th coordinate of the vector v.

Classically (see Norris [19] and for a synthetic exposition Bertoin [3]), this discrete time Markov chain is embedded in continuous time using a "Poissonization" of the time: given X^{DT}, one can recover $X = (X(t), t \geq 0)$ as follows. At time 0, $X(0) = X_0^{DT}$. For any vector $v \in \mathbb{R}^{m-1}$, define[2]

$$q(v) := \sum_{k=1}^{m-1} k l_k(v).$$

Let τ_1 be a random time exponentially distributed with parameter $q(X_0^{DT})$. For any time $t \in [0, \tau_1[$, let $X(t) = X(0) = X_0^{DT}$. At time τ_1, X jumps from $v = X(0)$ to $v + w_k$ with probability given by formula (3.1). More generally, let $\tau_0 = 0$ and for any $n \geq 1$, define the n-th jumping time τ_n by

$$\tau_n = \sum_{i=0}^{n-1} \frac{\epsilon_i}{q(X_i^{DT})},$$

where ϵ_i are independent random variables having the same exponential distribution with parameter 1. Let

$$X(t) = X(\tau_n) = X_n^{DT}, \quad \forall t \in [\tau_n, \tau_{n+1}[.$$

At time τ_{n+1}, X jumps from $v = X(\tau_n)$ to $v + w_k$ with probability given by formula (3.1). It is easy to see that this embedded process $X(t)$ is the same one as the branching process defined in Section 2.1.

When $X_0^{DT} = (1, 0, \ldots, 0)$, each $X_n^{(i)}, i = 1, \ldots, m-1$, can be seen as the number of nodes of type i in a tree T_n: the sequence $(T_n, n \geq 0)$ is a sequence of random m-ary trees which grow by successive insertions of keys in their leaves. Each node of these trees contains at most $m-1$ keys. Keys are i.i.d. random

[2] Note that $q = u_1$ where u_1 was defined by (2.4).

variables $x_i, i \geq 1$, with any diffusive distribution on the interval $[0,1]$. The tree $T_n, n \geq 0$, is recursively defined as follows: T_0 is reduced to an empty node-root; T_1 is reduced to a node-root which contains x_1, T_2 is reduced to a node-root which contains x_1 and x_2, ... , T_{m-1} has a node-root containing $x_1, \dots x_{m-1}$. As soon as the $(m-1)$-th key is inserted in the root, m empty subtrees of the root are created, corresponding from left to right to the m ordered intervals $I_1 =]0, x_{(1)}[, \dots, I_m =]x_{(m-1)}, 1[$, where $0 < x_{(1)} < \dots < x_{(m-1)} < 1$ are the ordered $m-1$ first keys. Each following key $x_m, \dots$ is recursively inserted in the subtree corresponding to the unique interval I_j to which it belongs. As soon as a node is saturated, m empty subtrees of this node are created.

For each $i = \{1, \dots, m-1\}$ and $n \geq 1$, $X_n^{(i)}$ is the number of nodes in T_n which contain $i-1$ keys (and i gaps or free places) after insertion of the n-th key; such nodes are named nodes of type i. We don't worry about the number of saturated nodes. The vector X_n^{DT} is called the composition vector of the m-ary search tree. It provides a model for the space requirement of the algorithm. One can refer to Mahmoud's book [17] for further details on search trees.

Notice that, in this dynamics, the insertion of a new key is *uniform* on the gaps; it can be read on the transition probabilities (3.1).

3.2 Embedding

The embedding properties are summarized in the following lemma.

Lemma 3.4

1) For any $n \geq 1$, the distribution of $\tau_n - \tau_{n-1}$ is $\mathcal{E}xp(n-1+N_0)$, where N_0 is the number of free places in $X(0)$: $N_0 = u_1(X(0))$.

2) the processes $(\tau_n)_{n\geq 1}$ and $(X(\tau_n))_{n\geq 1}$ are independent.

3) the processes $(X(\tau_n))_{n\geq 1}$ and $(X_n^{DT})_{n\geq 1}$ have the same distribution.

Proof. Part 1) is a consequence of the fact that the minimum of k independent $\mathcal{E}xp(1)$-distributed random variables is $\mathcal{E}xp(k)$-distributed, and that the total number of free places at time τ_n equals $n-1+N_0$.

Part 2) is the classical independence between the jump chain and the jump times in such Markov processes. The initial states and evolution rules of both Markov chains in discrete time and in continuous time are the same ones, so that Part 3) holds. □

Convention. From now on, thanks to Part 3) of Lemma 3.4, we will as usual suppose that the discrete-time process and the continuous-time process are built

on the *same* probability space on which

$$(X(\tau_n))_{n\geq 1} = (X_n^{DT})_{n\geq 1} \quad a.s.. \tag{3.2}$$

Remark. The important benefit we get with the embedding is the independence in the continuous-time process. This independence is the key point for the dislocation equations later on.

4 Asymptotics and martingale connection

In this section we present an order 2 expansion of the continuous time multitype branching process $(X(t))$ and its connection with the discrete time process (X_n^{DT}) defined in Section 3.1.

4.1 Asymptotics of the continuous time branching process

With the notations of Section 2 and especially the formulae (2.4), the random vector $X(t)$ admits the following order 2 expansion as t goes to infinity.

Theorem 4.5 (Asymptotics of continuous time process)
Suppose that $m \geq 27$. Then, as t tends to infinity,

$$X(t) = e^t \xi v_1\big(1+\varepsilon_1(t)\big) + 2\Re\left(e^{\lambda_2 t} W v_2\right)\big(1+\varepsilon_2(t)\big) + \boldsymbol{\varepsilon}_3(t), \tag{4.1}$$

where
• *ξ is a positive* Gamma-*distributed random variable with expectation $N_0 = u_1(X(0))$ (total weighted number of particles at time 0),*
• *W is a complex-valued random variable that admits moments of all orders $p \geq 1$ and whose expectation equals $u_2(X(0))$,*
• *the real-valued random variables $\varepsilon_1(t)$ and $\varepsilon_2(t)$ tend to 0 as t tends to $+\infty$, almost surely and in any L^p-space, $p \geq 1$,*
• *the random vector $\boldsymbol{\varepsilon}_3(t)$ is $o\left(e^{\lambda_2 t}\right)$ as t tends to $+\infty$, almost surely and in any L^p-space, $p \geq 1$.*

In the whole paper, W denotes our hero, namely the limit complex-valued random variable of the second order term in $X(t)$'s expansion, as in Theorem 4.5.

Remark 4.6 *One can reformulate (4.1) as follows:*
in any basis of the form $(v_1, \Re(v_2), \Im(v_2), \dots)$,
• $X(t)$*'s first coordinate has the expansion* $e^t\xi + o(e^t)$,
• $X(t)$*'s component on* $\mathrm{Span}_{\mathbb{R}}\left(\Re(v_2), \Im(v_2)\right)$ *has the expansion* $2\Re\left(e^{\lambda_2 t} W v_2\right) + o(e^{\lambda_2 t})$,
• *all other coordinates are* $o(e^{\lambda_2 t})$.

Proof of Theorem 4.5. Denote $\mathcal{A}$ the endomorphism of $\mathbb{R}^{m-1}$ having ${}^tR_G$ as matrix in the canonical basis. Let also $M(t) = \exp(-t\mathcal{A})X(t)$, for any $t \geq 0$. By standard arguments from multitype branching process theory, $(M(t))_{t\geq 0}$ is a vector-valued martingale. Since $m \geq 27$, the real part of λ_2 belongs to $]1/2, 1[$ so that the projected martingales $u_1(M(t))$ and $u_2(M(t))$ converge in L^p for any $p \geq 1$. For proofs of these results, see for example Athreya and Ney [1] or Janson [12] (especially Lemma 10.2 of Janson's paper for the L^p-boundedness, X being here an *irreducible* process in the sense of [12]). The random variables ξ and W are respectively defined by

$$\left\{ \begin{array}{l} \xi = \lim\limits_{t\to+\infty} u_1\left(e^{-t\mathcal{A}}X(t)\right) = \lim\limits_{t\to+\infty} e^{-t}u_1\left(X(t)\right), \\ \\ W = \lim\limits_{t\to+\infty} u_2\left(e^{-t\mathcal{A}}X(t)\right) = \lim\limits_{t\to+\infty} e^{-\lambda_2 t}u_2\left(X(t)\right). \end{array} \right. \tag{4.2}$$

An alternative proof of the L^p convergence can be made using the techniques of [21], as developed in [5] for two-colour urn processes. In particular, ξ's distribution is attained by explicit computation of its moments: for any nonnegative integer p, an elementary computation shows directly from (2.2) that the (so-called *reduced*) polynomial

$$Q := u_1\left(u_1+1\right)\left(u_1+2\right)\dots\left(u_1+p-1\right)$$

is an eigenvector for X's infinitesimal generator G, associated with the eigenvalue p. Thus $\mathbb{E}Q(X(t)) = e^{pt}Q(X(0))$ for any t. Besides, because of (4.2), $Q(X(t)) = e^{pt}\xi^p(1+o(1))$ as t tends to infinity, almost surely and in L^1. Finally, the last two equalities provide

$$\mathbb{E}\xi^p = Q(X(0)) = \frac{\Gamma\left(N_0+p\right)}{\Gamma\left(N_0\right)}.$$

This shows that the law of ξ is a Gamma distribution with parameter N_0 since a Gamma distribution is completely determined by its moments. The matrix R_G is diagonalizable on $\mathbb{C}$ since all roots of its characteristic polynomial are simple (see (2.3)). Extending notations (2.4), let $(u_\lambda)_{\lambda\in\mathrm{Sp}(\mathcal{A})}$ be a basis of linear forms,

each u_λ being an eigenform of G associated with the (complex) eigenvalue λ. Let also $(v_\lambda)_{\lambda\in\mathrm{Sp}(\mathcal{A})}$ be the dual basis of $(u_\lambda)_{\lambda\in\mathrm{Sp}(\mathcal{A})}$, each v_λ being thus a vector that satisfies $u_\lambda(v_\mu)=\delta_{\lambda,\mu}$ (Kronecker's notation). Note that one can choose $u_{\lambda_2}=u_2$ and, consequently, $v_{\lambda_2}=v_2$ (cf. notations (2.4)).
For any $t\geq 0$, split the spectral decomposition of the vector $X(t)$ with respect to G into four terms:

$$X(t)=\sum_{\lambda\in\mathrm{Sp}(\mathcal{A})}u_\lambda(X(t)).v_\lambda=X_1(t)+X_2(t)+X_3(t)+X_4(t),$$

where

$$\begin{cases} X_1(t)=u_1(X(t))v_1,\\[2ex] X_2(t)=u_{\lambda_2}(X(t))v_{\lambda_2}+u_{\overline{\lambda_2}}(X(t))v_{\overline{\lambda_2}},\\[2ex] X_3(t)=\displaystyle\sum_{1/2<\Re\lambda<\Re\lambda_2}u_\lambda(X(t))v_\lambda,\\[2ex] X_4(t)=\sum_{\Re\lambda<1/2}u_\lambda(X(t))v_\lambda.\end{cases}$$

Note that this partition of $\mathrm{Sp}(\mathcal{A})$ is valid because $\frac{1}{2}$ is not an eigenvalue of $\mathcal{A}$ as can be checked from (2.3). We deal separately with these four components of $X(t)$. Define $\boldsymbol{\varepsilon}_3$ by $\boldsymbol{\varepsilon}_3(t)=X_3(t)+X_4(t)$, for any $t\geq 0$.

- The formulae (4.2) provide directly the asymptotics

$$\begin{cases} X_1(t)=(e^t\xi+o(e^t))v_1,\\[2ex] X_2(t)=(e^{\lambda_2 t}W+o(e^{\lambda_2 t}))v_2+\overline{(e^{\lambda_2 t}W+o(e^{\lambda_2 t}))v_2}=2\Re\left((e^{\lambda_2 t}W+o(e^{\lambda_2 t}))v_2\right),\end{cases}$$

leading to the first two terms of the expansion (4.1).

- Suppose that λ is an eigenvalue of $\mathcal{A}$ such that $\frac{1}{2}<\Re\lambda<\Re\lambda_2$. Then, with the same general arguments as in the very beginning of the proof, it can be seen that

$$u_\lambda(M(t))=e^{-t\lambda}u_\lambda(X(t))$$

and that $(u_\lambda(M(t)))_{t\geq 0}$ is a convergent martingale, bounded in any L^p, $p\geq 1$. In particular, $u_\lambda(X(t))=o(e^{\lambda_2 t})$ as t tends to infinity, almost surely and in any L^p, $p\geq 1$. This shows that $X_3(t)$ is $o(e^{\lambda_2 t})$ when $t\to+\infty$.

- It remains to deal with the small eigenvalues, namely with all λ such that $\Re\lambda<\frac{1}{2}$.

Lemma 4.7 *Suppose that λ is an eigenvalue such that $\Re\lambda < \frac{1}{2}$ and let $\eta > 0$. Then, $e^{-(\frac{1}{2}+\eta)t}u_\lambda(X(t))$ is bounded almost surely and in any L^p-space, $p \geq 1$.*

The proof of this lemma is given just hereafter. Therefore, if $\Re\lambda < \frac{1}{2}$, then

$$e^{-\lambda_2 t}u_\lambda(X(t)) = e^{(1/2+\eta-\lambda_2)t}\left[e^{-(\frac{1}{2}+\eta)t}u_\lambda(X(t))\right] \underset{t\to\infty}{\longrightarrow} 0$$

almost surely as soon as $0 < \eta < \Re\lambda_2 - \frac{1}{2}$. Such η exist because $\Re\lambda_2 > \frac{1}{2}$. This shows that $X_4(t)$ is $o(e^{\lambda_2 t})$ when $t \to +\infty$. The same argument holds for the L^p convergence, making the proof complete. □

Proof of Lemma 4.7. The main idea consists in taking advantage of the following fact: when t belongs to the interval $[\tau_n, \tau_{n+1}[$, the vector $X(t)$ remains equal to X_n^{DT}. This being considered, we make use of the moment bounds of the discrete time process that can be found in [21] (Theorem 3.4 (1)): when $\Re\lambda < \frac{1}{2}$,

$$\forall p \geq 1, \forall\varepsilon > 0, \quad \mathbb{E}|u_\lambda(X_n^{DT})|^p = O\left(n^{p(\frac{1}{2}+\varepsilon)}\right), \qquad n \to +\infty. \tag{4.3}$$

- Almost sure bound: we prove that

$$\lim_{C\to+\infty} \mathbb{P}\left(\exists t > 0,\ e^{-(\frac{1}{2}+\eta)t}\left|u_\lambda(X(t))\right| > C\right) = 0, \tag{4.4}$$

which suffices to get the almost sure boundedness. Let $C > 0$, $\eta > 0$ and let λ be an eigenvalue such that $\Re\lambda < \frac{1}{2}$. The jump time τ_n tends almost surely to $+\infty$ which is a classical result that can be deduced from Lemma 3.4, so that

$$\mathbb{P}\left(\exists t > 0,\ e^{-(\frac{1}{2}+\eta)t}|u_\lambda(X(t))| > C\right) \leq \sum_{n\geq 0}\mathbb{P}\left(\exists t \in [\tau_n, \tau_{n+1}[,\ e^{-(\frac{1}{2}+\eta)t}|u_\lambda(X(t))| > C\right).$$

Since $X(t) = X_n^{DT}$ for any $t \in [\tau_n, \tau_{n+1}[$, this leads to

$$\mathbb{P}\left(\exists t > 0,\ e^{-(\frac{1}{2}+\eta)t}|u_\lambda(X(t))| > C\right) \leq \sum_{n\geq 0}\mathbb{P}\left(\left|u_\lambda\left(X_n^{DT}\right)\right| > Ce^{(\frac{1}{2}+\eta)\tau_n}\right)$$

Conditioning with respect to τ_n, using Markov inequality and the fact that τ_n and X_n^{DT} are independent, one gets successively, for any $p \geq 1$:

$$\begin{aligned}\mathbb{P}\left(\exists t > 0,\ e^{-(\frac{1}{2}+\eta)t}|u_\lambda(X(t))| > C\right) &\leq \sum_{n\geq 0}\mathbb{E}\left(\mathbb{P}\left(\left|u_\lambda(X_n^{DT})\right| > Ce^{(\frac{1}{2}+\eta)\tau_n}\middle|\tau_n\right)\right)\\ &\leq \sum_{n\geq 0}\mathbb{E}\left(\frac{\mathbb{E}\left|u_\lambda\left(X_n^{DT}\right)\right|^p}{C^p e^{p(\frac{1}{2}+\eta)\tau_n}}\right)\\ &= \frac{1}{C^p}\sum_{n\geq 0}\mathbb{E}\left|u_\lambda\left(X_n^{DT}\right)\right|^p\mathbb{E}\left(e^{-p(\frac{1}{2}+\eta)\tau_n}\right).\end{aligned}$$

The density of the n-th jump time τ_n is the function

$$u \in \mathbb{R} \longmapsto ne^{-u}\left(1-e^{-u}\right)^{n-1}\mathbf{1}_{\mathbb{R}_+}(u),$$

so that its Laplace transform can be elementarily computed: for any $s \geq 0$,

$$\mathbb{E}(e^{-s\tau_n}) = \frac{n!\Gamma(s+1)}{\Gamma(s+1+n)} \sim \Gamma(s+1)n^{-s}, \qquad n \to +\infty.$$

Together with (4.3), this leads to: $\forall \eta > 0, \forall \varepsilon > 0$, $\forall p \geq 1$,

$$\mathbb{E}\left|u_\lambda\left(X_n^{DT}\right)\right|^p \mathbb{E}\left(e^{-p(\frac{1}{2}+\eta)\tau_n}\right) = O\left(\frac{1}{n^{p(\eta-\varepsilon)}}\right), \qquad n \to +\infty$$

which is the general term of a convergent series as soon as one takes $\varepsilon < \eta$ and $p > \frac{1}{\eta-\varepsilon}$. Finally, letting C tend to infinity shows (4.4).

- Bound in L^p-space: let $p \geq 1$ and $t > 0$. Then,

$$\left\|e^{-(\frac{1}{2}+\eta)t}u_\lambda\left(X(t)\right)\right\|_p^p = e^{-(\frac{1}{2}+\eta)pt}\mathbb{E}\left|u_\lambda\left(X(t)\right)\right|^p.$$

Using the relation with the discrete time process $(X_n^{DT})_n$, one has successively

$$\begin{aligned}
\left\|e^{-(\frac{1}{2}+\eta)t}u_\lambda\left(X(t)\right)\right\|_p^p &= e^{-(\frac{1}{2}+\eta)pt}\sum_{n\geq 0}\mathbb{E}\Big(\mathbf{1}_{\tau_n\leq t<\tau_{n+1}}\left|u_\lambda\left(X(t)\right)\right|^p\Big)\\
&= e^{-(\frac{1}{2}+\eta)pt}\sum_{n\geq 0}\mathbb{E}\Big(\mathbf{1}_{\tau_n\leq t<\tau_{n+1}}\left|u_\lambda\left(X_n^{DT}\right)\right|^p\Big)\\
&= e^{-(\frac{1}{2}+\eta)pt}\sum_{n\geq 0}\mathbb{E}\Big(\mathbf{1}_{\tau_n\leq t<\tau_{n+1}}\Big)\mathbb{E}\Big(\left|u_\lambda\left(X_n^{DT}\right)\right|^p\Big),
\end{aligned}$$

where the last equality holds due to the independence between τ_n and X_n^{DT}. Besides, τ_n and $\tau_{n+1}-\tau_n$ are independent and $\tau_{n+1}-\tau_n$ is $\mathcal{E}xp(n+N_0)$-distributed (see (3.4)), so that, using the density of τ_n written above, one gets

$$\begin{aligned}
\mathbb{E}\Big(\mathbf{1}_{\tau_n\leq t<\tau_{n+1}}\Big) &= \mathbb{E}\Big(\mathbf{1}_{t\geq\tau_n}\mathbb{E}\Big(\mathbf{1}_{\tau_{n+1}-\tau_n\geq t-\tau_n}\big|\tau_n\Big)\Big)\\
&= \mathbb{E}\Big(\mathbf{1}_{t\geq\tau_n}e^{-(n+N_0)(t-\tau_n)}\Big)\\
&= \int_0^t e^{-(n+N_0)(t-u)}ne^{-u}\left(1-e^{-u}\right)^{n-1}du\\
&\leq ne^{-(n+1)t}\int_0^t\left(e^u-1\right)^{n-1}e^u du = \left(1-e^{-t}\right)^n e^{-t}.
\end{aligned}$$

Thus,

$$\left\| e^{-(\frac{1}{2}+\eta)t} u_\lambda \left(X(t)\right) \right\|_p^p \leq e^{-t} e^{-(\frac{1}{2}+\eta)pt} \sum_{n\geq 0} \left(1-e^{-t}\right)^n \mathbb{E}\Big(\left|u_\lambda \left(X_n^{DT}\right)\right|^p \Big).$$

Let now $\varepsilon > 0$. On one hand, (4.3) implies that

$$\mathbb{E}\Big(\left|u_\lambda \left(X_n^{DT}\right)\right|^p \Big) = O\left(n^{p(\frac{1}{2}+\varepsilon)}\right).$$

On the other hand, Stirling's formula applied to generalized binomial coefficients yields classically that for any $\alpha \in \mathbb{C}$,

$$[z^n](1-z)^{-\alpha-1} = \frac{n^\alpha}{\Gamma(\alpha+1)} \left(1 + O\left(\frac{1}{n}\right)\right),$$

where the notation $[z^n]A(z)$ means the coefficient of z^n in the power expansion of $A(z)$ at the origin. Consequently,

$$\mathbb{E}\Big(\left|u_\lambda \left(X_n^{DT}\right)\right|^p \Big) = O\left([z^n](1-z)^{-1-p(\frac{1}{2}+\varepsilon)}\right).$$

This implies that for any $\varepsilon > 0$, there exists a constant C_ε such that for any $t > 0$,

$$\left\| e^{-(\frac{1}{2}+\eta)t} u_\lambda \left(X(t)\right) \right\|_p^p \leq C_\varepsilon e^{-t} e^{-(\frac{1}{2}+\eta)pt} \left(1 - \left(1-e^{-t}\right)\right)^{-1-p(\frac{1}{2}+\varepsilon)} = C_\varepsilon e^{-pt(\eta-\varepsilon)}.$$

It suffices to take $\varepsilon = \eta/2$ to conclude that the L^p-norm of $e^{-(\frac{1}{2}+\eta)t} u_\lambda \left(X(t)\right)$ is bounded above. $\square$

Remark 4.8 *The distribution of W is infinitely divisible, because it is the limit of infinitely divisible ones, obtained by scaling and projection of infinitely divisible ones. Indeed, in finite time, for any $x_0 \in \mathbb{R}^{m-1}$, denote by $(X_{x_0}(t), t \geq 0)$ the process $(X(t), t \geq 0)$ defined in Section 2.1 starting from initial state x_0. By the branching property*

$$X_{x_0}(t) \stackrel{\mathcal{L}}{=} [n] X_{\frac{x_0}{n}}(t),$$

where the notation $[n]X$ denotes the sum of n independent copies of the random variable X. This fact has already been noticed by Janson ([12], proof of Theorem 3.9).

4.2 Martingale connection

In this subsection, we use the embedding equality (3.2) to deduce connections between the asymptotic behaviours of X_n^{DT} when $n \to +\infty$ and $X(t)$ when $t \to +\infty$.

For $m \geq 27$, it has been proved in [4] and [20] that

$$X_n^{DT} = nv_1 + \Re(n^{\lambda_2} W^{DT} v_2) + o(n^{\sigma_2}) \quad \text{a.s. and in } L^p,\ \ \forall p \geq 1, \tag{4.5}$$

where v_1, v_2 are the deterministic vectors defined in (2.4), W^{DT} is a complex-valued martingale limit, and the moments of W^{DT} can be recursively calculated.

Proposition 4.9 *The following two assertions hold:*

$$\lim_{n\to+\infty} ne^{-\tau_n} = \xi \ \ \textit{a.s. and in } \ L^p,\ \forall p \geq 1, \tag{4.6}$$

$$W = \xi^{\lambda_2}\ W^{DT} \quad \textit{a.s. with } \ \xi \ \textit{and } W^{DT} \ \textit{independent.} \tag{4.7}$$

The equality (4.7) is commonly referred to as "martingale connection".

Proof. We first prove (4.6). Applying the first projection to the embedding equality (3.2), we obtain that

$$u_1(X(\tau_n)) = u_1(X_n^{DT})) \ \ a.s.,$$

where u_1 has been defined in (2.4). This is the total number of free places at time τ_n, and is equal to $n-1+N_0 = n(1+o(1))$. Therefore, by (4.2) and the fact that the splitting times τ_n tend almost surely to $+\infty$ when n goes to $+\infty$, we have

$$\xi = \lim_{t\to+\infty} e^{-t} u_1(X(t)) = \lim_{n\to+\infty} ne^{-\tau_n} \quad a.s..$$

This gives (4.6).

We then prove (4.7). Applying the second projection to the embedding equality (3.2) we obtain

$$u_2(X(\tau_n)) = u_2(X_n^{DT})) \ \ a.s.,$$

where u_2 has been defined in (2.4). Using again (4.2) and the fact that τ_n goes to $+\infty$ when n goes to $+\infty$, we get

$$W = \lim_{t\to+\infty} e^{-\lambda_2 t} u_2(X(t)) = \lim_{n\to+\infty} e^{-\lambda_2 \tau_n} u_2(X_n^{DT}).$$

Therefore (4.7) follows from (4.6) and the asymptotics in discrete time given in (4.5). □

Remark 4.10 *Fill and Kapur ([9]) proved that W^{DT} is the unique solution in the space of probability distributions with a given mean and finite second absolute moment of the fixed point equation*

$$Z \stackrel{\mathcal{L}}{=} \sum_{k=1}^{m} (V_k)^{\lambda_2} Z^{(k)}, \tag{4.8}$$

where the V_k are the spaces in the statistical order of $(m-1)$ i.i.d. random variables uniformly distributed on $[0,1]$. Because $\mathbb{R}$-valued stable distributions are solutions of the fixed point equation (4.8) when λ is a real number, it is somehow natural to ask whether a λ-stable distribution is a solution of Eq. (4.8) for a complex number λ. By λ-stable we mean operator-stable when the operator is given by a two dimensional matrix $\lambda = \sigma + i\tau = \begin{pmatrix} \sigma & -\tau \\ \tau & \sigma \end{pmatrix}$ as introduced by Sharpe [24]. It is known since Hudson et al. [11] that a λ-stable distribution has infinite moments of order p for $p > 1/\Re(\lambda)$. Consequently, neither W nor W^{DT} (which have moments of any order) can be stable distributions.

5 A distributional equation

In this section we derive a distributional equation satisfied by the limit variable of the continuous-time branching process with an appropriate norming. We shall see that this equation characterizes the limit distribution.

5.1 Vectorial finite time dislocation equations

Let us write dislocation equations for the continuous-time branching process at finite time t. We write $X_j(t)$ for $X(t)$ when the process starts from $X(0) = e_j$, where e_j denotes the j-th vector of the canonical basis of $\mathbb{R}^{m-1}$ (whose j-th component is 1 and all the others are 0). This means that the process starts from an ancestor of type j.

Notice that the first splitting time τ_1 changes of distribution depending on the ancestor's type; denote by $\tau_{(j)}, j = 1, \ldots, m-1$, the first splitting time when the process starts from $X(0) = e_j$. Thus $\tau_{(j)}$ is $\mathcal{E}xp(j)$ distributed.

The branching property applied at the first splitting time gives:

$$\forall t > \tau_1, \quad \begin{cases} X_1(t) \overset{\mathcal{L}}{=} X_2(t - \tau_{(1)}), \\ X_2(t) \overset{\mathcal{L}}{=} X_3(t - \tau_{(2)}), \\ \dots \\ X_{m-2}(t) \overset{\mathcal{L}}{=} X_{m-1}(t - \tau_{(m-2)}), \\ X_{m-1}(t) \overset{\mathcal{L}}{=} [m]X_1(t - \tau_{(m-1)}), \end{cases} \tag{5.1}$$

where the notation $[m]X$ denotes the sum of m independent copies of the random variable X.

In the following, we denote

$$T = \tau_{(1)} + \cdots + \tau_{(m-1)}, \tag{5.2}$$

where the $\tau_{(j)}$ are independent of each other and each $\tau_{(j)}$ is $\mathcal{E}xp(j)$ distributed. Let us give some elementary properties of T that we shall need. It is easy to see that T has density

$$f_T(u) = (m-1)e^{-u}(1 - e^{-u})^{m-2}\mathbf{1}_{\mathbb{R}_+}(u), \quad u \in \mathbb{R}, \tag{5.3}$$

so that e^{-T} has a Beta distribution with parameters 1 and $m-1$. A straightforward change of variable under the integral shows that for any complex number λ such that $\Re(\lambda) > 0$,

$$\mathbb{E}e^{-\lambda T} = \int_0^{+\infty} e^{-\lambda u} f_T(u)du = (m-1)B(1+\lambda, m-1) \tag{5.4}$$

$$= \frac{(m-1)!}{\prod_{k=1}^{m-1}(\lambda + k)}, \tag{5.5}$$

where B denotes Euler's Beta function:

$$B(x,y) = \int_0^1 u^{x-1}(1-u)^{y-1}du = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}, \quad \Re x > 0, \Re y > 0.$$

In particular,

$$m\mathbb{E}|e^{-\lambda T}| \begin{cases} < 1 \text{ if } \Re(\lambda) > 1, \\ = 1 \text{ if } \Re(\lambda) = 1, \\ > 1 \text{ if } \Re(\lambda) < 1. \end{cases} \tag{5.6}$$

5.2 Distributional equation satisfied by the limit variable

After projections of variables $X_j(t)$ (the process starts from $X(0) = e_j$) with u_2, scaling with $e^{-\lambda_2 t}$ and taking the limit when t goes to infinity, we get the variables

$$W_j := \lim_{t \to +\infty} e^{-\lambda_2 t} u_2(X_j(t)),$$

so that the system (5.1) on $X_j(t)$ leads to the following system of distributional equations on W_j:

$$\begin{cases} W_1 \stackrel{\mathcal{L}}{=} e^{-\lambda_2 \tau_{(1)}} W_2, \\ W_2 \stackrel{\mathcal{L}}{=} e^{-\lambda_2 \tau_{(2)}} W_3, \\ \dots \\ W_{m-2} \stackrel{\mathcal{L}}{=} e^{-\lambda_2 \tau_{(m-2)}} W_{m-1}, \\ W_{m-1} \stackrel{\mathcal{L}}{=} e^{-\lambda_2 \tau_{(m-1)}} [m] W_1. \end{cases} \tag{5.7}$$

This shows that W_1 is a solution of the following fixed point equation:

$$Z \stackrel{\mathcal{L}}{=} e^{-\lambda_2 T} (Z^{(1)} + \dots + Z^{(m)}), \tag{5.8}$$

where $Z^{(i)}$ are independent copies of Z, which are also independent of T.

In terms of the Fourier transform

$$\varphi(t) := \mathbb{E} \exp\{i \langle t, Z \rangle\} = \mathbb{E} \exp\{i \Re(\bar{t} Z)\}, \quad t \in \mathbb{C},$$

where $\langle x, y \rangle = \Re(\overline{x} y) = \Re(x)\Re(y) + \Im(x)\Im(y)$, the equation (5.8) reads

$$\varphi(t) = \int_0^{+\infty} \varphi^m (t e^{-\overline{\lambda_2} u}) f_T(u) du, \quad t \in \mathbb{C}, \tag{5.9}$$

where f_T is defined by (5.3). Notice that this functional equation can also be written in a convolution form: if $\Phi(t) := \varphi(e^{\overline{\lambda_2} t})$ for any $t \in \mathbb{C}$, then Φ satisfies the following functional equation:

$$\Phi(t) = \int_0^{+\infty} \Phi^m (t - u) f_T(u) du, \quad t \in \mathbb{C}. \tag{5.10}$$

In the following sections, we prove that the distributional equation (5.8) characterizes the law of W_1 and we get several results on W_1: for example W_1 has

a density on the whole complex plane, and admits exponential moments. All these results appear as a particular case of a slightly more general situation given hereafter. From now on, for any complex number λ, consider the distributional equation

$$Z \overset{\mathcal{L}}{=} e^{-\lambda T}(Z^{(1)} + \cdots + Z^{(m)}), \tag{5.11}$$

where $Z^{(i)}$ are independent copies of Z, which are also independent of T. On Fourier transforms, it reads

$$\varphi(t) = \int_0^{+\infty} \varphi^m(te^{-\overline{\lambda}u}) f_T(u)du, \quad t \in \mathbb{C}, \tag{5.12}$$

where f_T is defined by (5.3).

Notice that when Z is a solution of the distributional equation (5.11), with finite and non zero first moment, then λ is a root of the polynomial function (2.3). In particular, λ is an algebraic number.

6 The smoothing transformation

A solution of the distributional equation (5.11) is a fixed point of the associated smoothing transformation defined hereafter by (6.1). Endowing a suitable space of probability measures with two distances, we prove that the smoothing transformation is a contraction for both metrics. This provides two alternative approaches for the study of Eq. (5.11) by contraction method. Using the Wasserstein distance as a first metric, we adapt the classical contraction method developed in [10], [22] and [23]. The second metric is defined in terms of Fourier transforms of measures; it provides a short proof of our result.

For any complex number C, let $\mathcal{M}_2(C)$ be the space of probability distributions on $\mathbb{C}$ admitting a second absolute moment and having C as expectation.

Let λ be a complex number. For any probability measure μ on $\mathbb{C}$, let

$$K\mu := \mathcal{L}\Big(e^{-\lambda T}(Z^{(1)} + \cdots + Z^{(m)})\Big), \tag{6.1}$$

where T is given by (5.2), $Z^{(i)}$ are independent random variables of law μ, which are also independent of T. Following Durrett and Liggett [8] who considered the case of real random variables, we call K the *smoothing transformation*. Note that K depends on m and λ.

Lemma 6.11 *If λ is a root of the characteristic polynomial (2.3) such that $\Re(\lambda) > -\frac{1}{2}$ and if C is any complex number, then K maps $\mathcal{M}_2(C)$ into itself.*

Proof. Since $\Re(\lambda) > -1$, the random variable $e^{-\lambda T}$ has an expectation. Furthermore, by (5.4), $m\mathbb{E}e^{-\lambda T} = 1$ as λ is a root of (2.3). This ensures the conservation of the expectation by K. Since $\Re(\lambda) > -\frac{1}{2}$, then $\mathbb{E}|e^{-\lambda T}|^2 < \infty$ and $K\mu$ admits a second absolute moment whenever μ does. Therefore $K\mu \in \mathcal{M}_2(C)$ whenever $\mu \in \mathcal{M}_2(C)$. □

Notice that a solution of Eq. (5.11) is a fixed point of K. We shall use the Banach fixed point theorem for two different metrics on $\mathcal{M}_2(C)$ to study the existence and uniqueness of solutions of Eq. (5.11).

6.1 Wassertein distance

Let d_2 be the Wasserstein distance on $\mathcal{M}_2(C)$ (see for instance Dudley [7]): for $\mu, \nu \in \mathcal{M}_2(C)$,

$$d_2(\mu, \nu) = \left(\min_{(X,Y)} \mathbb{E}\left(|X - Y|^2\right) \right)^{\frac{1}{2}},$$

where the minimum is taken over couples of random variables (X, Y) having respective marginal distributions μ and ν; the minimum is attained by the Kantorovich-Rubinstein Theorem – see for instance Dudley [7], p. 421. With this distance d_2, $\mathcal{M}_2(C)$ is a complete metric space.

Theorem 6.12 *Let $\lambda \in \mathbb{C}$ be a root of the characteristic polynomial (2.3) such that $\Re(\lambda) > \frac{1}{2}$, and let $C \in \mathbb{C}$. Then K is a contraction on the complete metric space $(\mathcal{M}_2(C), d_2)$, and the fixed point equation (5.11) has a unique solution Z in $\mathcal{M}_2(C)$.*

We now come back to the limit variable W_1 of m-ary search trees. Since $\mathbb{E}W_1 = 1$ and $\mathbb{E}|W_1|^2 < \infty$, the following corollary is a direct consequence of Theorem 6.12, applied for $\lambda = \lambda_2$.

Corollary 6.13 *The distribution of the limit complex random variable W_1 is the unique solution in the space $\mathcal{M}_2(1)$ of the fixed point equation (5.11).*

Proof of Theorem 6.12. We argue as in [10], [22] and [23] where real random variables were considered.

By the Banach fixed point theorem, it suffices to show the contraction property. Let $\mu, \nu \in \mathcal{M}_2(C)$. Let (X, Y) be a couple of complex-valued random variables such that $\mathcal{L}(X) = \mu$, $\mathcal{L}(Y) = \nu$ and $d_2(\mu, \nu) = \sqrt{\mathbb{E}|X - Y|^2}$. Let $(X_i, Y_i), i = 1, \ldots, m$ be m independent copies of the d_2-optimal couple (X, Y),

and T be a real random variable with density f_T defined by (5.3), independent from all (X_i, Y_i). Then,

$$\mathcal{L}(e^{-\lambda T}\sum_{i=1}^{m} X_i) = K\mu \quad \text{and} \quad \mathcal{L}(e^{-\lambda T}\sum_{i=1}^{m} Y_i) = K\nu,$$

so that

$$\begin{aligned}
d_2(K\mu, K\nu)^2 &\leq \mathbb{E}\left|\left(e^{-\lambda T}\sum_{i=1}^{m} X_i\right) - \left(e^{-\lambda T}\sum_{i=1}^{m} Y_i\right)\right|^2 \\
&= \mathbb{E}\left|e^{-\lambda T}\sum_{i=1}^{m}(X_i - Y_i)\right|^2 \\
&= \mathbb{E}\left|e^{-\lambda T}\right|^2 \mathbb{E}\left|\sum_{i=1}^{m}(X_i - Y_i)\right|^2 \\
&= \mathbb{E}\left|e^{-\lambda T}\right|^2 \left(\sum_{i=1}^{m}\mathbb{E}\left|X_i - Y_i\right|^2 + \sum_{i\neq j}\mathbb{E}\left(X_i - Y_i\right)\left(\overline{X_j} - \overline{Y_j}\right)\right) \\
&= m\mathbb{E}\left|e^{-2\lambda T}\right| \; d_2\left(\mu,\nu\right)^2.
\end{aligned}$$

Since $2\Re(\lambda) > 1$, we have $m\mathbb{E}\left|e^{-2\lambda T}\right| < 1$ (see (5.6)). Therefore K is a contraction on $\mathcal{M}_2(C)$ and the proof is complete.

□

6.2 Distance defined with Fourier transforms

We now give an alternative approach for the characterization of the limit distribution via Fourier analysis. We define another distance d_2^* on $\mathcal{M}_2(C)$ as follows. Take $\mu, \nu \in \mathcal{M}_2(C)$ and denote respectively φ and ψ their characteristic functions. By definition of $\mathcal{M}_2(C)$, both φ and ψ admit the expansion $\varphi(t) = 1 + i\langle t, C\rangle + O\left(|t|^2\right)$ when t tends to 0. Therefore, one can define $d_2^*(\mu,\nu)$ by

$$d_2^*(\mu,\nu) = \sup_{t\in\mathbb{C}\setminus\{0\}} \frac{|\varphi(t) - \psi(t)|}{|t|^2}.$$

Clearly, $d_2^*(\varphi,\psi) < \infty$, and d_2^* is a distance on $\mathcal{M}_2(C)$. It can be easily checked that $(\mathcal{M}_2(C), d_2^*)$ is a complete metric space.

The following result is a counterpart of Theorem 6.12. It gives an alternative proof for the existence and uniqueness of the solution of Eq. (5.11) in the class of probability measures on $\mathbb{C}$ with a given mean and finite second moments.

Theorem 6.14 *Let $\lambda \in \mathbb{C}$ be a root of the characteristic polynomial (2.3) such that $\Re(\lambda) > \frac{1}{2}$, and let $C \in \mathbb{C}$. Then K is a contraction on the complete metric space $(\mathcal{M}_2(C), d_2^*)$, and the fixed point equation (5.11) has a unique solution Z in $\mathcal{M}_2(C)$.*

Proof. Thanks to Banach fixed point theorem, it suffices to prove that K is a contraction on $\mathcal{M}_2(C)$ equipped with the metric d_2^*. Let $\mu, \nu \in \mathcal{M}_2(C)$ and let φ and ψ be their respective characteristic functions. An elementary computation shows that the Fourier transform of $K\mu$ is $t \mapsto \mathbb{E}\varphi^m(e^{-\overline{\lambda}T}t)$ with a corresponding formula for ν. We have $|\varphi| \leq 1$, $|\psi| \leq 1$, so that

$$\mathbb{E}|\varphi^m(e^{-\overline{\lambda}T}t) - \psi^m(e^{-\overline{\lambda}T}t)| \leq m\mathbb{E}|\varphi(e^{-\overline{\lambda}T}t) - \psi(e^{-\overline{\lambda}T}t)|.$$

Together with the inequality $|\varphi(z) - \psi(z)| \leq d_2^*(\mu, \nu)|z|^2$ applied to $z = e^{-\overline{\lambda}T}t$, this implies that

$$d_2^*(K\mu, K\nu) \ \leq \ m\mathbb{E}\left(e^{-2\Re(\lambda)T}\right) d_2^*(\mu, \nu).$$

Since $2\Re(\lambda) > 1$, we have $m\mathbb{E}\left(e^{-2\Re(\lambda)T}\right) < 1$ (see (5.6)). Therefore the above inequality shows that K is a contraction on $(\mathcal{M}_2(C), d_2^*)$. $\square$

Remark 6.15 *Denote $\mathcal{F}_2(C)$ the space of Fourier transforms of elements of $\mathcal{M}_2(C)$. When λ is a root of the characteristic polynomial (2.3) such that $\Re(\lambda) > \frac{1}{2}$, the smoothing transformation K can be identified as a map (also denoted by K) on $\mathcal{F}_2(C)$ given by*

$$(K\varphi)(t) := \mathbb{E}\varphi^m(e^{-\overline{\lambda}T}t), \qquad t \in \mathbb{C}. \tag{6.2}$$

The proof of Theorem 6.14 also shows that K is a contraction of $\mathcal{F}_2(C)$ for the metric (also denoted by d_2^*) defined on $\mathcal{F}_2(C)$ by

$$d_2^*(\varphi, \psi) := \sup_{t \neq 0} \frac{|\varphi(t) - \psi(t)|}{|t|^2}. \tag{6.3}$$

Let $\mathcal{D}_2(C)$ be the space of all continuous functions $\varphi : \mathbb{C} \to \mathbb{C}$ that admit an expansion $\varphi(t) = 1 + i\langle t, C\rangle + O\left(|t|^2\right)$ at 0 and such that $||\varphi||_\infty \leq 1$. Clearly, $\mathcal{D}_2(C)$ contains $\mathcal{F}_2(C)$. One can show that formula (6.2) defines a mapping from $\mathcal{D}_2(C)$ into itself and that K is a contraction for the metrics defined by (6.3). This provides a proof of existence and unicity of solutions of (5.12) on $\mathcal{D}_2(C)$.

Remark 6.16 *One can deal with the convolution equation (5.10) by arguments of the same vein. Similar computations show that this equation has a unique solution in the space* $\mathcal{E}_2(\lambda, C)$ *of continuous functions* $\Phi : \mathbb{C} \to \mathbb{C}$ *that admit an expansion* $\Phi(z) = 1 + i\langle e^{\lambda z}, C\rangle + O\left(e^{2\lambda z}\right)$ *when* $|z|$ *tends to* $+\infty$ *and such that* $||\Phi||_\infty \leq 1$*. This result appears one again as a consequence of Banach Theorem on* $\mathcal{E}_2(\lambda, C)$ *for the metric*

$$d(\Phi, \Psi) = \sup_{z\in\mathbb{C}} \left| \frac{\Phi(z) - \Psi(z)}{e^{2\lambda z}} \right|.$$

As a consequence, this shows in particular that the Fourier (complex) transform φ *of* W_1 *satisfies: for any* $w \in \mathbb{C}^*$ *and for any determination of the logarithm,*

$$\varphi(w) = \Phi\left(\frac{\log w}{\overline{\lambda_2}}\right),$$

where Φ *is the unique solution in* $\mathcal{E}_2(\overline{\lambda_2}, 1)$ *of Eq. (5.10). This result is the reversed version of the change of variable* $\Phi(z) = \varphi(e^{\overline{\lambda_2} z})$ *that led from (5.9) to (5.10).*

7 Density and support

In this section we prove results on the absolute continuity and on the support of solutions of the distributional equation (5.11) via Fourier analysis. As applications, we show that the distribution of the limit variable W_1 of the multitype branching process $(X(t))$ has always a density and that its support is the whole complex plane.

Theorem 7.17 *Let* λ *be a complex number such that* $\lambda \neq 1$ *and* $\Re(\lambda) > 0$*. Let* Z *be a complex-valued random variable solution of the distributional equation (5.11)*

$$Z \stackrel{\mathcal{L}}{=} e^{-\lambda T}(Z^{(1)} + \cdots + Z^{(m)}),$$

with $\mathbb{E}|Z| < \infty$ *and* $\mathbb{E}Z \neq 0$*. Then the following assertions hold:*

(i) the support of Z *is the whole complex plane* $\mathbb{C}$*;*

(ii) as $|t| \to \infty$*,* $\mathbb{E}e^{i\langle t, Z\rangle} = O(|t|^{-a})$*, for any* $a \in]0, \dfrac{1}{\Re(\lambda)}[$*;*

(iii) the distribution of Z *has a density with respect to the Lebesgue measure on* $\mathbb{C}$*.*

Remark 7.18 *When $\lambda = 1$, the distributional equation (5.11) becomes*

$$X \overset{\mathcal{L}}{=} e^{-T}(X^{(1)} + \cdots + X^{(m)}). \tag{7.1}$$

By Section 6, it admits a unique solution in the space $\mathcal{M}_2(C)$ of probability measures on $\mathbb{C}$, with a given mean C. Moreover, from the dislocation equations (5.1), a similar argument shows that

$$\xi := \lim_{t\to+\infty} e^{-t} u_1(X(t))$$

is a solution of this equation. By Theorem 4.5, ξ is Gamma-distributed. Therefore the unique solution of (7.1) in $\mathcal{M}_2(C)$ is $C\gamma$ where γ is Gamma(1)- distributed, and its support is the half line $C\mathbb{R}_+$.

The following corollary gives the main result for the limit variable W_1 of the multitype branching process. It is a direct consequence of Theorem 7.17 since $\mathbb{E}W_1 = 1$.

Corollary 7.19 *The distribution of W_1 admits a density with respect to the Lebesgue measure on $\mathbb{C}$, and its support is the whole complex plane $\mathbb{C}$. Moreover, as $|t| \to \infty$, $\mathbb{E}e^{i\langle t, W_1\rangle} = O(|t|^{-a})$ for each $a \in]0, \dfrac{1}{\Re(\lambda_2)}[$.*

The proof of Theorem 7.17 runs along the following lines. Let φ be the Fourier transform of any solution Z of (5.11). We prove that φ is in $L^2(\mathbb{C})$ because it is dominated by $|t|^{-\delta}$ for some $\delta > 1$ so that the inverse Fourier-Plancherel transform provides a square integrable density for Z. The guiding idea consists in adapting usual methods (developed in [14] and [15]) used for positive real-valued random variables to our complex-valued case thanks to the function defined for $r \geq 0$ by

$$\psi(r) = \max_{|t|=r} |\varphi(t)|.$$

From now on,

$$A = e^{-\lambda T}.$$

We proceed by a series of lemmas.

The first lemma concerns a property of the support of Z. For a complex-valued random variable Z and for a complex number z, by definition,

$$z \in \mathrm{Supp}(Z) \Longleftrightarrow \forall \varepsilon > 0, \mathbb{P}(|Z - z| \leq \varepsilon) > 0.$$

Lemma 7.20 *Let $z \in \mathbb{C}$. Then*

$$z \in \mathrm{Supp}(Z) \Longrightarrow D(0, |z|) \subseteq \mathrm{Supp}(Z),$$

where $D(0, |z|)$ denotes the open disc with center 0 *and radius $|z|$.*

PROOF. We first prove the following implication:

$$\big[a \in \mathrm{Supp}(A) \text{ and } z \in \mathrm{Supp}(Z)\big] \Longrightarrow maz \in \mathrm{Supp}(Z).$$

Indeed, let $\varepsilon > 0$, $a \in \mathrm{Supp}(A)$ and $z \in \mathrm{Supp}(Z)$. Let also $Z^{(1)}, \dots, Z^{(m)}$ be independent copies of Z. Then, with positive probability, $|A - a| \leq \varepsilon$ and $|Z^{(k)} - z| \leq \varepsilon$ for any k. Therefore, with positive probability,

$$\begin{aligned}\big|A(Z^{(1)} + \cdots + Z^{(m)}) - maz\big| &= \left|mz(A-a) + A\sum_{k=1}^{m}(Z^{(k)} - z)\right| \\ &\leq m\varepsilon|z| + (|a| + \varepsilon)\, m\varepsilon.\end{aligned}$$

The positive real ε being arbitrary, this shows that $maz \in \mathrm{Supp}\, A(Z^{(1)} + \cdots + Z^{(m)})$ which implies that $maz \in \mathrm{Supp}(Z)$ by Eq. (5.11), proving the claim.

Let $z \in \mathrm{Supp}(Z)$. Since $\mathrm{Supp}(T) = \mathbb{R}_+$ (see (5.3)), the claim implies that for any $t \geq 0$, $me^{-\lambda t}z \in \mathrm{Supp}(Z)$. Iterating this property of $\mathrm{Supp}(Z)$ shows that

$$\left\{m^n e^{-\lambda t} z,\ n \in \mathbb{N},\ t \in \mathbb{R}_+\right\} \subseteq \mathrm{Supp}(Z). \tag{7.2}$$

Since the support of a probability measure is a closed set, to show that $D(0, |z|) \subseteq \mathrm{Supp}(Z)$ it suffices to prove that $\left\{m^n e^{-\lambda t},\ n \in \mathbb{N},\ t \in \mathbb{R}_+\right\}$ is everywhere dense in the unit disc. Taking logarithm, we show hereunder that

$$\mathcal{G} := \mathbb{N}\log m + 2i\pi\mathbb{N} - \lambda\mathbb{R}_+$$

is everywhere dense in the half-strip

$$\mathcal{B} := \{x + iy,\ x < 0,\ \ -2\pi < y \leq 0\}$$

which implies the desired result.

Let σ and τ denote respectively the real and imaginary parts of λ. Remember that, as soon as Z is a solution of Eq.(5.11) such that $\mathbb{E}|Z| < \infty$ and $\mathbb{E}Z \neq 0$, then λ is a root of χ_{R_G} defined by (2.3); this implies that λ is an algebraic number, and so is σ/τ.

Let us prove that $\rho := i\pi/\log m$ is a transcendental number. In fact, if ρ were algebraic, then by the Gelfond-Schneider theorem[3], m^ρ would be a transcendental number; but this is impossible because $m^\rho = \exp(i\pi) = -1$. Therefore ρ is a transcendental number.

It follows that $2\pi\sigma/(\tau \log m)$ is not an algebraic number, hence not a rational number. So by a classical result,

$$\mathbb{N} \log m - \mathbb{N}\frac{2\pi\sigma}{\tau} \quad \text{is a dense subset of } \mathbb{R}.$$

Let $b = x + iy \in \mathcal{B}$ with $x < 0$ and $-2\pi < y \leq 0$, and let $\varepsilon > 0$. Approximating the real number $x - \frac{y\sigma}{\tau}$ by an element of $\mathbb{N} \log m - \mathbb{N}\frac{2\pi\sigma}{\tau}$, we take $n, k \in \mathbb{N}$ such that

$$\left| \left(n \log m - k\frac{2\pi\sigma}{\tau}\right) - \left(x - \frac{y\sigma}{\tau}\right) \right| \leq \varepsilon.$$

Therefore

$$|b - g| \leq \varepsilon, \text{ where } g = n \log m + 2ik\pi - t\lambda \in \mathcal{G} \text{ with } t = \frac{2k\pi - y}{\tau} \geq 0.$$

This completes the proof of Lemma 7.20. □

Lemma 7.20 leads to the following key property of ψ, which will imply that the characteristic function φ of Z satisfies the Cramér condition and then the Riemann-Lebesgue condition.

Lemma 7.21 $\forall r > 0, \psi(r) < 1$.

Proof. Obviously, $\psi(0) = 1$ and $\psi(r) \leq 1$ for any $r \geq 0$. Suppose that $r_0 > 0$ is such that $\psi(r_0) = 1$. Take thus $z_0 \in \mathbb{C}$ and $\theta_0 \in \mathbb{R}$ such that

$$|z_0| = r_0 \quad \text{and} \quad \mathbb{E}e^{i\langle z_0, Z\rangle} = e^{i\theta_0}.$$

The complex random variable $e^{i(\langle z_0, Z\rangle - \theta_0)}$ is of mean 1 and takes its values on the unit disc, so that it is almost surely equal to 1. This implies that $\mathrm{Supp}(Z)$ is contained in a set of countably many parallel lines of the complex plane. This contradicts Lemma 7.20 since such a set of lines is negligible with respect to the Lebesgue measure on $\mathbb{C}$. □

[3]The Gelfond-Schneider theorem states that if a and b are algebraic numbers with $a \neq 0, 1$ and if b is not a rational number, then any value of $a^b = \exp(b \log a)$ is a transcendental number.

Remark 7.22 *The preceding arguments show the following assertion: for any complex-valued random variable Z, if $|\mathbb{E}(e^{i\langle z_0,Z\rangle})| = 1$ for some $z_0 \in \mathbb{C} \setminus \{0\}$, then* $\mathrm{Supp}(Z) \subseteq a + b\mathbb{Z} + c\mathbb{R}$ *for some $a, b, c \in \mathbb{C}$ (a set of countably many parallel lines). The algebraicity of λ that leads to the proof of Lemma 7.20 can thus be seen as a nonlattice assumption on the fixed point equation (5.11).*

We now prove that the characteristic function φ satisfies the Riemann-Lesbesgue condition.

Lemma 7.23 $\lim\limits_{r\to+\infty} \psi(r) = 0.$

Proof. We argue as in the proof of Theorem 3.1 of [14] or Lemma 3.1 of [15]. Notice that from the distributional equation (5.12) we have

$$\psi(r) \le \mathbb{E}(\psi^m(r|A|)). \tag{7.3}$$

• We first prove that $\limsup\limits_{r\to+\infty} \psi(r) = 0$ or 1. By Fatou's lemma,

$$\limsup_{r\to+\infty} \psi(r) \le \mathbb{E} \limsup_{r\to+\infty} \psi^m(r|A|) = \left(\limsup_{r\to+\infty} \psi(r)\right)^m,$$

the last equality coming from $\mathbb{P}(|A| > 0) = 1$. So the real number $l := \limsup_r \psi(r)$ satisfies both $l \le 1$ and $l \le l^m$; this implies that $l = 0$ or 1.

• Suppose that $\limsup_r \psi(r) = 1$. By Lemma 7.21, $\psi(1) < 1$. For any $\varepsilon \in]0, 1 - \psi(1)[$, define

$$\begin{cases} r_1(\varepsilon) = \max\{r \in]0,1[, \psi(r) = 1-\varepsilon\}, \\ \\ r_2(\varepsilon) = \min\{r > 1, \psi(r) = 1-\varepsilon\}. \end{cases}$$

These quantities are well defined because $\psi(0) = 1$ and ψ is continuous. Then $\psi(r_1(\varepsilon)) = \psi(r_2(\varepsilon)) = 1-\varepsilon$ and for any $r \in [r_1(\varepsilon), r_2(\varepsilon)]$, $\psi(r) \le 1-\varepsilon$.

Let us prove that $r_1(\varepsilon)$ goes to 0 when ε tends to 0. Take any limit point ρ of $r_1(\varepsilon)$. Since ψ is continuous, $\psi(\rho) = 1-\varepsilon$ which implies by Lemma 7.21 that $\rho = 0$: the only possible limit point is 0.

By (7.3), we have

$$\psi(r) \le \mathbb{E}\psi(r|A|).$$

Iterating this inequality we see that for all $n \ge 1$,

$$\psi(r) \le \mathbb{E}\psi(r|A_1|\dots|A_n|),$$

where $(|A_i|)_{i\geq 1}$ are independent copies of $|A|$. With the notation

$$\lambda_n(r,\varepsilon) := \mathbb{P}\Big(r_1(\varepsilon) < r|A_1|\dots|A_n| \leq r_2(\varepsilon)\Big),$$

we have for any $r>0$

$$\psi(r) \leq (1-\varepsilon)\lambda_n(r,\varepsilon) + 1 - \lambda_n(r,\varepsilon) = 1-\varepsilon\lambda_n(r,\varepsilon).$$

Again by (5.12),

$$1-\varepsilon = \psi(r_2(\varepsilon)) \leq \mathbb{E}\psi^m\Big(r_2(\varepsilon)|A|\Big) \leq \mathbb{E}\left(1-\varepsilon\lambda_n\Big(r_2(\varepsilon)|A|,\varepsilon\Big)\right)^m.$$

In other words

$$\frac{\mathbb{E}\left(1-\left(1-\varepsilon\lambda_n\Big(r_2(\varepsilon)|A|,\varepsilon\Big)\right)^m\right)}{\varepsilon} \leq 1. \tag{7.4}$$

We are going to pass to the limit in the above ratio when ε tends to 0. Rewrite

$$\lambda_n\Big(r_2(\varepsilon)|A|,\varepsilon\Big) = \mathbb{P}\left(\frac{r_1(\varepsilon)}{r_2(\varepsilon)} < |A||A_1|\dots|A_n| \leq 1\right),$$

and remember that $r_1(\varepsilon)\leq 1\leq r_2(\varepsilon)$ and that $r_1(\varepsilon)$ goes to 0 when ε tends to 0, so that $\frac{r_1(\varepsilon)}{r_2(\varepsilon)} \leq \frac{r_1(\varepsilon)}{1}$ goes to 0 when ε tends to 0. Consequently,

$$\lambda_n\Big(r_2(\varepsilon)|A|,\varepsilon\Big) \underset{\varepsilon\to 0}{\longrightarrow} \mathbb{P}\Big(0\leq |A||A_1|\dots|A_n|\leq 1\Big) = \mu_n(|A|) \quad a.s.,$$

where, for any $x>0$,

$$\mu_n(x) := \mathbb{P}\Big(x|A_1|\dots|A_n|\leq 1\Big).$$

Therefore

$$\frac{1-\left(1-\varepsilon\lambda_n\Big(r_2(\varepsilon)|A|,\varepsilon\Big)\right)^m}{\varepsilon} \underset{\varepsilon\to 0}{\longrightarrow} m\mu_n(|A|) \quad a.s..$$

The above ratio is a function of ε, uniformly bounded on the compact set $[0,1-\psi(1)]$, so that by dominated convergence and (7.4),

$$\frac{\mathbb{E}\left(1-\left(1-\varepsilon\lambda_n\Big(r_2(\varepsilon)|A|,\varepsilon\Big)\right)^m\right)}{\varepsilon} \underset{\varepsilon\to 0}{\longrightarrow} m\mathbb{E}\mu_n(|A|) \leq 1. \tag{7.5}$$

Besides, by Markov inequality

$$1-\mu_n(x) \leq x\mathbb{E}(|A_1|\dots|A_n|) = x\left(\mathbb{E}|A|\right)^n.$$

Since $\Re(\lambda) > 0$, $\mathbb{E}|A| = \mathbb{E}|e^{-\lambda T}| < 1$ (see (5.4)), which implies that $\lim_{n\to\infty} \mu_n(x) = 1$, so that

$$\lim_{n\to\infty} \mathbb{E}\mu_n(|A|) = 1$$

by dominated convergence. This contradicts (7.5) because $m \geq 2$. $\square$

We shall need an information about the decay rate of $\varphi(t)$, of the form $\varphi(t) = O(|t|^{-\delta})$ for some $\delta > 0$ when $|t| \to \infty$. To this end, we shall use the following Gronwall-type technical Lemma of [14] (see also Lemma 3.2 in [15]).

Lemma 7.24 *[14] Let $\psi : \mathbb{R}_+ \to \mathbb{R}_+$ be a bounded function and let B be a positive random variable such that for some constants $p \in]0,1[, a > 0, C \geq 0$ and for all $r > 0$,*

$$\psi(r) \leq p\mathbb{E}\psi(Br) + Cr^{-a}.$$

If $p\mathbb{E}(B^{-a}) < 1$, then $\psi(r) = O(r^{-a})$ as $r \to \infty$.

This is Lemma 4.1 of [14]. It can be proved as follows. Let $\{B_i\}$ be independent copies of B. Then by induction, for all $n \geq 1$ and all $r > 0$,

$$\psi(r) \leq p^n \mathbb{E}\psi(B_1 ... B_n r) + Cr^{-a}[1 + p\mathbb{E}(B^{-a}) + ... + (p\mathbb{E}(B^{-a}))^{n-1}].$$

Letting $n \to \infty$ we see that for all $r > 0$,

$$\psi(r) \leq Cr^{-a}/[1 - p\mathbb{E}(B^{-a})].$$

Lemma 7.25 *For all $a \in]0, \dfrac{1}{\Re(\lambda)}[$, as $r \to \infty$,*

$$\psi(r) = O(r^{-a}).$$

Proof. We have already seen from the distributional equation (5.12) that

$$\psi(r) \leq \mathbb{E}(\psi^m(r|A|)),$$

where $A = e^{-\lambda T}$. By Lemma 7.23, for any $\varepsilon > 0$, there is some $r_\varepsilon > 0$ such that $\forall r \geq r_\varepsilon$, $\psi(r) \leq \varepsilon$. So

$$\psi(r) \leq \varepsilon^{m-1}\mathbb{E}\psi(r|A|) + \mathbb{P}(r|A| \leq r_\epsilon).$$

Therefore by Markov inequality, for $a \in]0, \dfrac{1}{\Re(\lambda)}[$,

$$\psi(r) \leq \varepsilon^{m-1}\mathbb{E}\psi(r|A|) + r^{-a}(r_\varepsilon)^a \mathbb{E}(|A|^{-a}).$$

By (5.4), $\mathbb{E}(|A|^{-a}) = (m-1)B(1-a\Re(\lambda), m-1) < \infty$. Taking $\varepsilon > 0$ small enough such that $\varepsilon^{m-1}\mathbb{E}(|A|^{-a}) < 1$, we see that the desired result follows from Lemma 7.24 and the preceding inequality on $\psi(r)$. □

We can now finish the proof of Theorem 7.17.

Proof of Theorem 7.17. Part (i) of the theorem comes from two facts as shown in the following.

On the one hand, by Lemma 7.20, as soon as $z \in \mathbb{C}$ is a point in the support of Z, we have $D(0,|z|) \subseteq \mathrm{Supp}(Z)$, where $D(0,|z|)$ denotes the open disc with center 0 and radius $|z|$.

On the other hand, the support of Z is unbounded. Indeed, as in (7.2), at the beginning of the proof of Lemma 7.20, as soon as $z \in \mathbb{C} \setminus \{0\}$ is a point in the support of Z, for any $t > 0$ and for any $n \in \mathbb{N}$, $m^n e^{-\lambda t} z$ is in the support of Z.

For Part (ii), notice that by Lemma 7.25, for all $a \in]0, \frac{1}{\Re(\lambda)}[$,

$$\varphi(t) = O(t^{-a}) \qquad \text{as } |t| \to \infty. \tag{7.6}$$

Since $\mathbb{E}Z \neq 0$, by Eq. (5.12) we obtain $m\mathbb{E}e^{-\lambda T} = 1$, hence $m\mathbb{E}|e^{-\lambda T}| = m\mathbb{E}e^{-\Re(\lambda)T} > 1$ as soon as $\Im(\lambda) \neq 0$. Notice that if $\Im(\lambda) = 0$, then $\lambda = 1$ by the equation $m\mathbb{E}e^{-\lambda T} = 1$. So the hypotheses $\lambda \neq 1$ and $\mathbb{E}Z \neq 0$ imply that $\Im(\lambda) \neq 0$ and $\Re(\lambda) < 1$ (cf. (5.6)). It follows that (7.6) holds for some $a > 1$, so that the Fourier transform φ of Z is in L^2. Therefore by the inversion formula of Fourier-Plancherel transform, the distribution of Z has a density in L^2 with respect to the Lebesgue measure on $\mathbb{C}$. This ends the proof of Theorem 7.17. □

Remark 7.26 *In fact we have the following more general result. Let λ be an complex number with $\sigma := \Re(\lambda) > 0$, $\tau := \Im(\lambda) \neq 0$ and satisfying the arithmetical condition:*

$$\frac{\pi\sigma}{\tau \log m} \notin \mathbb{Q}$$

and let Z be a nontrivial solution of Eq. (5.11) (with or without first moment). Then the distribution of Z is absolutely continuous with respect to the Lebesgue measure on $\mathbb{C}$, and its support is the whole complex plane $\mathbb{C}$.

To see the conclusions of Remark 7.26, we can argue as follows. In the general case where the expectation of Z may not exist, Lemma 7.20 still holds thanks to the arithmetical condition. The remaining of the proof is the same, except at the end, where $\Re(\lambda) > 1$ is no more ensured. Nevertheless we have an additional

argument by iteration. Iterating the distributional equation (5.11), we obtain for $n \geq 1$,

$$Z \stackrel{\mathcal{L}}{=} \sum_{u_1 \ldots u_n \in \{1,\ldots,m\}^n} A A_{u_1} \ldots A_{u_1 \ldots u_{n-1}} Z^{(u_1 \ldots u_n)},$$

where $A = e^{-\lambda T}$, A_u are independent copies of A (indexed by finite sequences of integers u), $Z^{(u)}$ are independent copies of Z, the two families $\{A_u\}$ and $\{Z^{(u)}\}$ are also independent of each other; by convention, $A_{u_1} \ldots A_{u_1 \ldots u_{n-1}}$ is taken to be 1 when $n = 1$. It is convenient to rewrite this equation in the form

$$Z \stackrel{\mathcal{L}}{=} \sum_{j=1}^{m^n} Y_j Z^{(j)}, \tag{7.7}$$

where $Z^{(j)}$ are independent copies of Z which are also independent of $\{Y_j\}$. For fixed $y = (y_j : 1 \leq j \leq m^n)$ with $\prod_{j=1}^{m^n} y_j \neq 0$, by Lemma 7.25, for $a \in]0, 1/\Re(\lambda)[$ and some constant $c > 0$,

$$\begin{aligned} \left| \mathbb{E} \exp \left(i \langle t, \sum_{j=1}^{m^n} y_j Z^{(j)} \rangle \right) \right| &\leq \prod_{j=1}^{m^n} c |t y_j|^{-a} \\ &= C(y) |t|^{-m^n a}, \end{aligned}$$

where $C(y) = \prod_{j=1}^{m^n} c |y_j|^{-a} > 0$ does not depend on t. Let $n \geq 1$ be large enough such that $m^n a > 1$. Then $\sum_{j=1}^{m^n} y_j Z^{(j)}$ is absolute continuous (with respect to the Lebesgue measure on $\mathbb{C}$) as its Fourier transform is square integrable on $\mathbb{C}$. This implies that for each Borel set B of $\mathbb{C}$ with Lebesgue measure 0, we have

$$\mathbb{P}\left(\sum_{j=1}^{m^n} y_j Z^{(j)} \in B\right) = 0.$$

It follows from Eq. 7.7 (by conditioning on (Y_j)) that $\mathbb{P}(Z \in B) = 0$.

8 Exponential moments and Laplace series

In this section we consider a solution Z of Eq. (5.11) and we show that its exponential moments exist in a neighborhood of 0, so that the moment exponential generating series of Z defines an analytic function in a neighbourhood of the origin. We show that this function satisfies a very simple differential equation.

Theorem 8.27 *Let $\lambda \in \mathbb{C}$ be a root of the characteristic polynomial (2.3) with $\Re(\lambda) > 1/2$ and let Z be a solution of Eq. (5.11). There exist some constants $C > 0$ and $\varepsilon > 0$ such that for all $t \in \mathbb{C}$ with $|t| \le \varepsilon$,*

$$\mathbb{E}e^{\langle t,Z\rangle} \le e^{\Re(t)+C|t|^2} \quad \text{and} \quad \mathbb{E}e^{|tZ|} \le 4e^{|t|+2C|t|^2}. \tag{8.1}$$

To prove this theorem, we use Mandelbrot's cascades in the complex setting (see Barral et al. [2] for independent interest about complex Mandelbrot's cascades). We still denote $A = e^{-\lambda T}$. Then $m\mathbb{E}A = 1$ because λ is a root of the characteristic polynomial (2.3) and $m\mathbb{E}|A|^2 < 1$ because $\Re(\lambda) > 1/2$ (see (5.6)). Let $A_u, u \in U$ be independent copies of A, indexed by all finite sequences of integers

$$u = u_1 \dots u_n \in U := \bigcup_{n\ge 1} \{1, 2, \dots, m\}^n$$

and set $Y_0 = 1$, $Y_1 = mA$ and for $n \ge 2$,

$$Y_n = \sum_{u_1\dots u_{n-1} \in \{1,\dots,m\}^{n-1}} mAA_{u_1}A_{u_1u_2}\dots A_{u_1\dots u_{n-1}}.$$

As $m\mathbb{E}A = 1$, $(Y_n)_n$ is a martingale. This martingale has been studied by many authors in the real random variable case, especially in the context of Mandelbrot's cascades, see for example [15] and the references therein. It can be easily seen that

$$Y_{n+1} = A\sum_{i=1}^{m} Y_{n,i} \tag{8.2}$$

where $Y_{n,i}$ for $1 \le i \le m$ are independent of each other and independent of A and each has the same distribution as Y_n. Therefore for $n \ge 1$, Y_n is square-integrable and

$$\operatorname{Var} Y_{n+1} = (\mathbb{E}|A|^2m^2 - 1) + m\mathbb{E}|A|^2 \operatorname{Var} Y_n,$$

where $\operatorname{Var} X = \mathbb{E}\left(|X - \mathbb{E}X|^2\right)$ denotes the variance of X. Since $m\mathbb{E}|A|^2 < 1$, the martingale $(Y_n)_n$ is bounded in L^2, so that the following result holds.

Lemma 8.28 *Let λ be a root of the characteristic polynomial (2.3) with $\Re(\lambda) > 1/2$. Then, when $n \to +\infty$,*

$$Y_n \to Y_\infty \ \textit{a.s. and in } L^2,$$

where Y_∞ is a (complex-valued) random variable with variance

$$\operatorname{Var}(Y_\infty) = \frac{\mathbb{E}|A|^2m^2 - 1}{1 - m\mathbb{E}|A|^2}.$$

Notice that, passing to the limit in (8.2) gives a new proof of the existence of a solution Z of Eq. (5.11) with $\mathbb{E}Z = 1$ and finite second moment whenever $\Re(\lambda) > 1/2$. From Section 6, we have the unicity of solution of this equation so that Theorem 8.27 is proved as soon as it holds for Y_∞.

Lemma 8.29 *Under the condition of Lemma 8.28, there exist some constants $C > 0$ and $\varepsilon > 0$ such that for all $t \in \mathbb{C}$ with $|t| \leq \varepsilon$, we have*

$$\mathbb{E}e^{\langle t, Y_\infty \rangle} \leq e^{\Re(t) + C|t|^2}. \tag{8.3}$$

Proof. As in [22] and [16] (where a similar problem for real random variables was considered), we use an induction argument. Notice that by Eq. (8.2), writing

$$\varphi_n(t) := \mathbb{E}e^{\langle t, Y_n \rangle}, \qquad t \in \mathbb{C}, n \geq 0,$$

we have

$$\varphi_{n+1}(t) = \mathbb{E}\varphi_n^m(\overline{A}t), \qquad t \in \mathbb{C}. \tag{8.4}$$

We shall prove that there exist some constants $C > 0$ and $\varepsilon > 0$ such that for all $n \geq 0$ and all $t \in \mathbb{C}$ with $|t| \leq \varepsilon$, we have

$$\varphi_n(t) \leq e^{\Re(t) + C|t|^2}. \tag{8.5}$$

Let us prove (8.5) by induction. The inequality holds clearly for $n = 0$ since $\varphi_0(t) = e^{\Re(t)}$. Assume that it holds for some $n \geq 0$ and all $t \in \mathbb{C}$ with $|t| \leq \varepsilon$. Then writing $A = A_1 + iA_2$ ($A_i \in \mathbb{R}$), using $|A| \leq 1$ and Eq. (8.4), we have for $t = t_1 + it_2$ ($t_i \in \mathbb{R}$) with $|t| \leq \varepsilon$,

$$\begin{aligned} \varphi_{n+1}(t) &\leq \mathbb{E}\exp\{m(A_1 t_1 + A_2 t_2 + C|A|^2(t_1^2 + t_2^2))\} \\ &= e^{\Re(t) + C|t|^2} g(t_1, t_2), \end{aligned} \tag{8.6}$$

where $g(t_1, t_2) = \mathbb{E}e^{h(t_1, t_2)}$ with

$$h(t_1, t_2) = (mA_1 - 1)t_1 + mA_2 t_2 + C(m|A|^2 - 1)(t_1^2 + t_2^2).$$

Notice that $g(0, 0) = 1$. It remains to prove that (0,0) is a local maximum of g. Clearly,

$$\begin{aligned} \frac{\partial g}{\partial t_i} &= \mathbb{E}e^h\left[\frac{\partial h}{\partial t_i}\right], \qquad i = 1, 2 \\ \frac{\partial^2 g}{\partial t_i} &= \mathbb{E}e^h\left[(\frac{\partial h}{\partial t_i})^2 + \frac{\partial^2 h}{\partial t_i}\right], \qquad i = 1, 2 \\ \frac{\partial^2 g}{\partial t_1 \partial t_2} &= \mathbb{E}e^h\left[\frac{\partial h}{\partial t_1}\frac{\partial h}{\partial t_1} + \frac{\partial^2 h}{\partial t_1 \partial t_2}\right]. \end{aligned}$$

Notice that, a.s.

$$\frac{\partial h}{\partial t_1}(0,0) = (mA_1 - 1), \qquad \frac{\partial h}{\partial t_2}(0,0) = mA_2,$$

$$\frac{\partial^2 h}{\partial t_1 \partial t_2}(0,0) = 0, \qquad \frac{\partial^2 h}{\partial t_i}(0,0) = 2C(m|A|^2 - 1), \qquad i = 1, 2.$$

Recall that $m\mathbb{E}A = 1$, so that $m\mathbb{E}A_1 = 1$ and $m\mathbb{E}A_2 = 0$; hence

$$\frac{\partial g}{\partial t_1}(0,0) = \mathbb{E}(mA_1 - 1) = 0, \qquad \frac{\partial g}{\partial t_2}(0,0) = \mathbb{E}(mA_2) = 0,$$

so that $(0,0)$ is a critical point of g. Moreover,

$$\begin{aligned}
\frac{\partial^2 g}{\partial t_1}(0,0) &= \mathbb{E}\left[(mA_1 - 1)^2 + 2C(m|A|^2 - 1)\right], \\
\frac{\partial^2 g}{\partial t_2}(0,0) &= \mathbb{E}\left[(mA_2)^2 + 2C(m|A|^2 - 1)\right], \\
\frac{\partial^2 g}{\partial t_1 \partial t_2}(0,0) &= \mathbb{E}(mA_1 - 1)(mA_2).
\end{aligned}$$

As $\mathbb{E}(m|A|^2 - 1) < 0$ (recall that $\Re(\lambda) > 1/2$), it follows that the Hessian matrix at $(0,0)$ is definite negative for $C > 0$ large enough which implies that $g(0,0)$ is a local maximum of g. So for $\varepsilon > 0$ small enough, $g(t_1, t_2) \le g(0,0) = 1$ for all $t = t_1 + it_2$ with $|t| \le \varepsilon$. Hence by (8.6), for such ε and C which do not depend on n, (8.5) holds for $n + 1$. Therefore, by induction, it holds for all $n \ge 0$.

Letting $n \to \infty$ in (8.5), we see that inequality (8.3) holds by Fatou's lemma. □

Proof of Theorem 8.27. By the unicity of solution of Eq. (5.11), $\mathcal{L}(Z) = \mathcal{L}(Y_\infty)$. So by Lemma 8.29, there are some constants $C > 0$ and $\varepsilon > 0$ such that the first inequality of (8.1) holds. To show the second one, notice that $|t||\Re(Z)| + |t||\Im(Z)|$ takes one of the four values $\pm|t|\Re(Z) \pm |t|\Im(Z)$ (according to the signs of $\Re(Z)$ and $\Im(Z)$), so that a.s.

$$\begin{aligned}
e^{|tZ|} &\le e^{|t||\Re(Z)| + |t||\Im(Z)|} \\
&\le e^{|t|\Re(Z) + |t|\Im(Z)} + e^{|t|\Re(Z) - |t|\Im(Z)} + e^{-|t|\Re(Z) + |t|\Im(Z)} + e^{-|t|\Re(Z) - |t|\Im(Z)}.
\end{aligned}$$

Taking expectation in both sides, and noticing that $\pm|t|\Re(Z) \pm |t|\Im(Z) = \langle(\pm 1 \pm i)|t|, Z\rangle$, we see that the second inequality in (8.1) follows from the first one. □

Suppose that Z is any solution of Eq. (5.11) under the assumptions of Theorem 8.27. The second inequality (8.1) shows that the exponential generating series of absolute moments of Z has a positive radius of convergence so that the formal Laplace series

$$L(z) := \sum_{p\geq 0} \frac{\mathbb{E}Z^p}{p!} z^p$$

defines an analytic function in a neighbourhood of the origin. One can also write $L(z) = \mathbb{E}e^{zZ}$ when $|z|$ is sufficiently small.

Let's come back to the dislocation equations (5.7) satisfied by the limit variables $W_1, \dots, W_m$. These variables admit finite (absolute) moments at any order. For any $k \in \{1, \dots, m\}$, let L_k be the formal Laplace series defined by

$$L_k(z) := \sum_{p\geq 0} \frac{\mathbb{E}\left(W_k^p\right)}{p!} z^p.$$

The dislocation equations (5.7) imply recursive relations on W_k's moments. Developing these relations with the multinomial formula implies that L_k satisfy the formal differential system

$$\begin{cases} \forall k \in \{1, \dots, m-2\},\ L_k(z) + \dfrac{\lambda_2}{k} z L_k'(z) = L_{k+1}(z), \\ L_{m-1}(z) + \dfrac{\lambda_2}{m-1} z L_{m-1}'(z) = \left(L_1(z)\right)^m, \end{cases} \tag{8.7}$$

with boundary conditions

$$\begin{cases} L_k(0) = 1,\ 1 \leq k \leq m-1, \\ L_k'(0) = \mathbb{E}(W_k) = u_2\left(X_k(0)\right) = \dbinom{\lambda_2 + k - 1}{k-1}. \end{cases}$$

Since W_1 satisfies the assumptions of Theorem 8.27, the series L_1 has a positive radius of convergence as shown above. Therefore, the same holds for all L_k because of the system (8.7) so that the L_k define, near the origin, analytic functions related by (8.7).

Let ρ be any complex $(m-1)$-th root of $(-1)^m(m-1)!$. For any $k \in \{1, \dots, m\}$, define

$$G_k(z) := (-1)^k \rho (k-1)! \frac{L_k\left(z^{-\lambda_2}\right)}{z^k},$$

where $z^{-\lambda_2}$ denotes any determination of the logarithm. For sufficiently large $|z|$, this formula defines an analytic function on a slit plane. Reporting in formula (8.7) shows that the functions G_k satisfy the simple differential system

$$\begin{cases} \forall k \in \{1, \dots, m-2\},\ G_k' = G_{k+1}, \\ G_{m-1}' = G_1^m. \end{cases}$$

In particular, G_1 is solution of the differential equation $y^{(m-1)} = y^m$. We sum up these results in the following statement.

Theorem 8.30 *Let W_1 be the complex-valued limit distribution for the multitype branching process of m-ary search trees as defined in Section 5.2. Then:*
(i) the Laplace series $L_1(z) = \mathbb{E}\left(e^{zW_1}\right)$ has a positive radius of convergence;
(ii) for any determination of the logarithm, the function

$$z \mapsto -\frac{\rho}{z} L_1\left(z^{-\lambda_2}\right),$$

is a solution of the differential equation

$$y^{(m-1)} = y^m. \tag{8.8}$$

Remark 8.31 *As can be straightforwardly checked, the function $y_\kappa(z) := \frac{\kappa}{1-z}$ is a solution of Eq. (8.8) when the complex number κ satisfies $\kappa^{m-1} = (m-1)!$. Nonetheless, G_1 is not a function of this form.*

Indeed, since $L_1(w) = 1 + w + o(w)$ in a neighbourhood of the origin, G_1 admits the expansion

$$G_1(z) = -\frac{\rho}{z} - \frac{\rho}{z^{1+\lambda_2}} + o\left(\frac{1}{z^{1+\lambda_2}}\right),$$

while y_κ satisfies

$$\frac{\kappa}{1-z} = -\frac{\kappa}{z} - \frac{\kappa}{z^2} + o\left(\frac{1}{z^2}\right).$$

One concludes by unicity of (complex) power expansions, because $\lambda_2 \neq 1$.

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